Transferring P_∞-structures

(krátká verze)

A_∞ (strongly homotopy associative) recall.

A_∞-algebra: $A = (\underbrace{A, \partial}_{\text{chain complex}}, \mu_2, \mu_3, \dots)$; $\mu_n : A^{\otimes n} \to A$

multiplication associative up to chain homotopy μ_3

deg $n-2$ linear map

A_∞-morphism: $(A, \partial, \mu) \xrightarrow{F = (f_1, f_2, \dots)} (B, \partial, \nu)$

collective notation for $\mu_2, \mu_3, \dots$

$f_s : A^{\otimes s} \to B$ deg $s-1$

chain map that is $\mu_2 - \nu_2$ homomorphism up to chain homotopy f_2

A_∞-homotopy

$$\begin{tikzcd} (A, \partial, \mu) \arrow[r, bend left, "F = (f_1, f_2, \dots)"] \arrow[r, bend right, "G = (g_1, g_2, \dots)"'] \arrow[r, phantom, "H = (h_1, h_2, \dots)"] & (B, \partial, \nu) \end{tikzcd}$$

$h_t : A^{\otimes t} \to B$

$\deg(h_t) = t$

chain homotopy $f_1 \overset{h_1}{\sim} g_1$

mesures deviation from h_1 to be derivation homotopy

In [M. Markl: Transferring A_∞ (strongly homotopy associative) structures. Rend. Circ. Mat. Palermo (2), Suppl. 79 (2006), 139 - 151 (preprint arxiv: math/0401007] I proved the following that holds over arbitrary comut. ring, including $\mathbb{Z}$.

Theorem (Markl 2006) Situation

chain maps

$$\begin{tikzcd} (A,\partial,\mu) \arrow[loop left, "h"] \arrow[r, bend left, "f"] & (B,\partial) \arrow[l, bend left, "g"] \end{tikzcd}$$

A_∞-structure

initial data

chain homotopy
$gf \overset{h}{\sim} 1\!\!1_A$

$$\Downarrow$$

$\exists$ A_∞ structure (B,∂,ν) on (B,∂) and extension

$$\begin{tikzcd} (A,\partial,\mu) \arrow[loop left, "{H=(h,h_2,\dots)}"] \arrow[r, bend left, "{F=(f,f_2,\dots)}"] & (B,\partial,\nu) \arrow[l, bend left, "{G=(g,g_2,\dots)}"] \end{tikzcd}$$

in red: Data of the "Situation"

NB: neither $fg = 1\!\!1_B$ nor infamous side conditions $hh=0$, $hg=0$, $fh=0$ required!

Everything given by explicit formulas.

Question: Does an analog of my theorem hold for other important sh algebras such as

$L_\infty, C_\infty, Pois_\infty, Gerst_\infty, \dots$

Need framework, manageable and still general enough quadratic Koszul operads over field of char 0.

Accepted definition: P - quad. Koszul $\Rightarrow$

$\boxed{P_\infty\text{-algebra}}$ is $(C_{P^i}(\uparrow A), \delta)$

cofree conilpotent P^i-coalgebra

coderivation, $\deg(\delta) = -1$, $\delta^2 = 0$

Examples:

$P = Ass$, $P^i = Ass^c$, $A_\infty = (T^c(\uparrow A), \delta)$

$P = Lie$, $P^i = Com^c$, $L_\infty = (\Lambda^c(\uparrow A), \delta)$

$P = Com$, $P^i = Lie^c$, $C_\infty = (\mathbb{L}^c(\uparrow A), \delta)$

$\boxed{P_\infty\text{-morphisms:}}$ morphisms of dg coalgebras

$\boxed{\text{Homotopy between } P_\infty\text{-morphisms:}}$ several equivalent definitions, e.g. one induced by CMC structure of P^i-comilpotent coalgebras

Due to Bruno Vallette. No time go to details.

Standing assumption: $\mathcal{P}$ quadratic Koszul / field of char 0

My approach of 1996: $\mathcal{P}_\infty$ algebra = algebra over the minimal model: $(\mathcal{P}, \partial = 0) \xleftarrow{\rho} \mathcal{M} = (F(X), \partial)$

operad morphism $H(\rho)$ is iso

free operad generated by X

minimality means no linear and const. part

Theorem [Markl 1996] $\mathcal{P}(0) = 0$, $\mathcal{P}(1) \cong \mathbb{k}$
$\Rightarrow \mathcal{M}\ \exists$, unique up to isomorphism.

satisfyed for quadratic Koszul

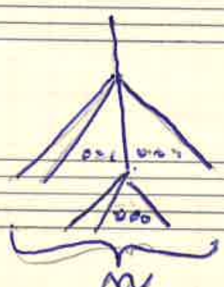
Example: $\mathcal{P} = \mathcal{A}ss$

$$\mathcal{A}ss = \frac{F(\lambda)}{(\lambda - \lambda)} \xleftarrow{\rho} (F(\lambda, \lambda, \lambda, \dots), \partial) = \mathcal{M}$$

gives $\mu_2, \mu_3, \mu_4, \dots$

$\partial(\lambda) = 0$ generator for multiplication

$\partial(\lambda) = \lambda - \lambda$ kills associativity

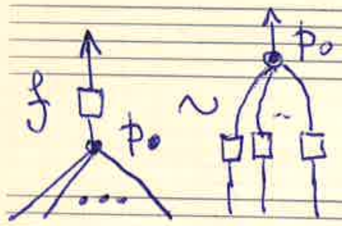
$$\partial(\underbrace{\lambda}_{n}) = \sum \pm$$ 

$\rho(\lambda) = \lambda$, $\rho(\lambda) = \rho(\lambda) = \dots = 0$.

Fact: $\mathcal{P}$ quadratic Koszul $\Rightarrow$ above def. equiv. to previous.

Morphism of P_∞ [Markl 2000] given $\mathcal{P}$, $\exists\ P_{\bullet\to\circ}$

two-coloured $P_{\bullet\to\circ} := \dfrac{P_\bullet \sqcup P_\circ \sqcup F(\bullet\!-\!\square\!\to\circ)}{\sim}$



$P_{\bullet\to\circ}$ algebras are diagrams $X \xrightarrow{f} Y$

P_algebras

morphism of P-algs

Definition: $\boxed{P_\infty\text{-morphism}}$ is an algebra over the minimal model $\mathcal{M}_{\bullet\to\circ}$ of $P_{\bullet\to\circ}$

Theorem. $P(0)=0,\ P(1)=\mathbb{K} \Rightarrow \mathcal{M}_{\bullet\to\circ}\ \exists$, unique up to iso.
If P is quadratic Koszul (case of Ass, Lie, Com, Pois, Gerst...) $\Rightarrow$ my Definition gives the same as standard one.

Homotopy of P_∞-morphisms

assuming $P(0)=0,\ P(1)=\mathbb{K}$.

Theorem. $\exists$ "minimal" model $P_{\bullet\to\circ} \xleftarrow{s} \mathcal{M}_{\bullet\rightrightarrows\circ}$
such that $\mathcal{M}_{\bullet\rightrightarrows\circ}(1) = (F(f,g,h),\ \partial h = f-g)$

arity 1 piece

represent chain maps $f,g : X \to Y$ and homotopy $h: f\sim g$

To distinguish $\mathcal{M}_{\bullet\rightrightarrows\circ}$ from $\mathcal{M}_{\bullet\to\circ}$ I will write it as

$$P_{\bullet\rightrightarrows\circ} \longleftarrow \mathcal{M}_{\bullet\rightrightarrows\circ}$$

isomorphic to $P_{\bullet\to\circ}$

operad in algebras diagrams $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, $f=g$

P_algebras

Consider the diagram

$$\begin{tikzcd}
\mathcal{M}_{\bullet\to\circ} \arrow[d, "i_u"'] \arrow[rrd, bend left, "F"] & & \\
\mathcal{M}_{\bullet \rightleftarrows \circ} \arrow[rr, dashed, "H"] & & \mathrm{End}_{X,Y} \\
\mathcal{M}_{\bullet\to\circ} \arrow[u, "i_e"] \arrow[rru, bend right, "G"'] & &
\end{tikzcd}$$

i_u (i_e) is inclusion to the "upper" ("bottom") arrow

F, G are P_∞-morphisms as defined before

Definition P_∞-morphisms F & G are P_∞-homotopic if $\exists H$ in diagram above. H is a P_∞-homotopy

Theorem [Dotsenko, Poncin] P quadratic Koszul (such as $\mathcal{A}ss$, $\mathcal{L}ie$, $\mathcal{C}om$, $\mathcal{P}oiss$, ..) $\Rightarrow$ my definition equivalent to all standard and commonly accepted ones.

Proof of P_∞-transfer theorem – how operadic approach gives (deceivingly) simple proof. P quadratic Koszul as before.

$$P_S : P_\bullet \sqcup \frac{F(\bullet \xrightarrow{f} \circ, \circ \xrightarrow{g} \bullet)}{(gf = 1\!\!1_\bullet)}. \quad P_S\text{-algebra is } \begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

P-algebra

chain maps s.t. $gf = 1\!\!1$

$\mathcal{P}_{\underline{S}} \xleftarrow{\rho_{\underline{S}}} \mathcal{M}_{\underline{S}}$ "minimal" model

By our philosophy $\mathcal{P} \longmapsto \mathcal{P}_\infty$, $gf = 1\!\!1 \longmapsto gf \overset{h}{\sim} 1\!\!1$

resolves to

Thus $\mathcal{M}_{\underline{S}}$-algebra is

$$h \circlearrowleft (A, \partial, \mu) \underset{g}{\overset{f}{\rightleftarrows}} (B, \partial)$$

initial situation

$\mathcal{P}_S$ operad for $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ f, g - $\mathcal{P}$-morphisms s.t. $gf = 1\!\!1_X$

P-algebras

$\mathcal{P}_S \xleftarrow{\rho_S} \mathcal{M}_S$ "minimal" model

$\mathcal{P} \longmapsto \mathcal{P}_\infty$, $f, g \longmapsto \mathcal{P}_\infty$ maps F, G

$gf = 1\!\!1 \longmapsto \mathcal{P}_\infty$ homotopy between GF and $1\!\!1$

So $\mathcal{M}_S$-algebra is $H \circlearrowleft (A, \partial, \mu) \underset{G}{\overset{F}{\rightleftarrows}} (B, \partial, \nu)$

the required extension

∃ of transfers will follow from the existence of Diagram

$$\begin{tikzcd} \mathcal{M}_S \arrow[r, "\Lambda"] \arrow[d, "\rho_S"'] \arrow[rr, bend left=40, "\alpha\Lambda"] & \mathcal{M}_{\underline{S}} \arrow[d, "\rho_{\underline{S}}"] \arrow[r, "\alpha"] & \mathrm{End}_{A,B} \\ P_S \arrow[r, "\lambda"] & P_{\underline{S}} & \end{tikzcd}$$

α ... initial data

$\alpha\Lambda$ .. required extension

How I get it ?

λ explicit: $\lambda(f) = f$, $\lambda(g) = g$, $\lambda(p_\bullet) = p_\bullet$, $\lambda(p_0) = f p_\bullet g^{\otimes n}$ for $p_0 \in P_0(M)$

Λ is the lift in

$$\begin{tikzcd} * \arrow[r, hook] \arrow[d, "\text{Cof}"'] & \mathcal{M}_{\underline{S}} \arrow[d, "\rho_{\underline{S}}", "\text{Fib} \cap \text{WE}"'] \\ \mathcal{M}_S \arrow[r, "\lambda\rho_S"'] \arrow[ur, dashed, "\Lambda"] & P_{\underline{S}} \end{tikzcd}$$

initial operad

Fib ∩ WE

Cof

∎

Advantage: explains principle why true

Disadvantage: Does not give formulas
needs advanced tech